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1987 J. Phys. A: Math. Gen. 20 2827

(http://iopscience.iop.org/0305-4470/20/10/026)

J. Phys. A: Math. Gen. 20 (1987) 2827-2832. Printed in the UK

Casimir effect, the gauge problem and seagull terms†

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Received 29 August 1986

Abstract. Much recent interest in the classic Casimir effect problem has urged us to treat this phenomenon carefully by giving special attention to the gauge problem, which has been almost enitrely swept under the carpet in the past, and to seagull terms. By quantising in the so-called Coulomb gauge, where the dynamical degrees of freedom are apparent and the quantisation problem is unambiguous, the expression for the Casimir effect is derived by explicitly taking into account seagull terms. Although a physical quantity ought to be gauge independent a specific gauge, nevertheless, has to be chosen for its evaluation.

1. Introduction

Much attention has been given recently (e.g. Gonzáles 1986, 1985, DeRaad 1985, De Raad and Milton 1981, Davies and Unwin 1981, Kennedy *et al* 1980, and references therein) (see also Balian and Duplantier 1978, Brown and Maclay 1969, Schwinger 1975, Schwinger *et al* 1978 with the latter two in the source theory scheme, for some earlier work) to the classic Casimir effect (Casimir 1948) which has been experimentally observed (e.g. Sabisky and Anderson 1973, Israelachvili and Tabor 1972, Tabor and Winterton 1969, Sparnay 1958, Deriagin and Abrikosova 1957). The basic physical phenomenon, as observed in the simplest situation, is the appearance of an attractive force arising between two neutral macroscopic conducting plates in a vacuum, which is attributed to the non-zero vacuum stress of the Maxwellian field. The emphasis in the abovementioned recent theoretical studies was mainly on improvements of computational techniques, but the gauge problem was, unfortunately, almost entirely neglected. This led to treating the Maxwellian field components as if they are independent or to study the problem for a scalar field and the final result was them multiplied by two to take into account the photon spin. The purpose of this paper is to remedy this problem, in the full context of quantum field theory, by quantising the Maxwellian field at the outset in the so-called Coulomb gauge where the degrees of freedom are apparent from the very beginning and a consistent formulation (Manoukian 1986, see also Fradkin and Tyutin 1970) of the gauge problem may be carried out by imposing commutation relations only on the independent dynamical degrees of freedom. We also pay special attention to seagull terms arising from the action principle involving constrained dynamics. Although a physical quantity ought to be gauge independent a specific gauge, nevertheless, has to be chosen for its evaluation.

† Work supported by the Department of National Defence Award under CRAD No 3610-637:F4122.

0305-4470/87/102827+06$02.50

2. Derivation

The Lagrangian density for the Maxwellian field in the presence of an external current J_μ may be written as

$$L = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu \tag{1}$$

and we impose the constraint

$$\partial_k A^k = 0 \qquad k = 1, 2, 3. \tag{2}$$

The latter equation allows one to solve (Fradkin and Tyutin 1970, Manoukian 1986) for A^3:

$$A^3 = -\partial_3^{-1}(\partial_i A^i) \qquad i = 1, 2. \tag{3}$$

By treating A^0, A^1, A^2 as basic field components in (1) (later we will see in (7) that A^0 is a dependent field) one readily obtains the field equations:

$$\partial_k F^{k0} = -J^0 \qquad k = 1, 2, 3 \tag{4}$$

$$\partial_\mu F^{\mu i} = \partial_3^{-1}\partial^i(\partial_\nu F^{\nu 3} + J^3) - J^i \qquad i = 1, 2. \tag{5}$$

Upon eliminating the expression in the square brackets on the right-hand side of (5) and combining the resulting equation with (4) we obtain

$$\partial_\mu F^{\mu\nu} = -\left(g^{\nu\sigma} - g^{\nu k}\frac{\partial_k \partial^\sigma}{\vec{\partial}^2}\right)J_\sigma \tag{6}$$

and the latter is consistent with $\partial_\nu\partial_\mu F^{\mu\nu} = 0$ for *all* J_σ. That is, J_σ need not be conserved and we may vary J_σ arbitrarily to generate Green functions. This point cannot be overemphasised.

The canonical momenta as obtained from (1) are

$$\pi(A^0) = 0 \qquad \pi(A^i) \equiv \pi^i = \partial_3^{-1}\partial^0 F^{i3} \qquad i = 1, 2 \tag{7}$$

and hence A^0 is a dependent field so defined since its conjugate momentum vanishes. The basic canonical commutation relations imposed are then

$$\delta(x^0 - x'^0)[A^i(x'), \pi^j(x)] = \mathrm{i}\delta^{ij}\delta(x - x') \qquad i, j = 1, 2. \tag{8}$$

From (4) and (7) we may also write (see also Fradkin and Tyutin 1970):

$$F^{0k} = -\left(g^{ki} - \frac{\partial^k\partial^i}{\vec{\partial}^2}\right)\pi^i + \frac{\partial^k}{\vec{\partial}^2}J^0 \tag{9}$$

$k = 1, 2, 3$; $i = 1, 2$. Equations (8) and (9) then *lead* to the basic commutation relation

$$\delta(x^0 - x'^0)[A^m(x'), F^{0k}(x)] = \mathrm{i}\left(\frac{\partial^k\partial^m}{\vec{\partial}^2} - \delta^{km}\right)\delta(x - x') \tag{10}$$

k, $m = 1, 2, 3$, where we have also made use of (3).

The equation of motion of the electric field $E^k = F^{0k}$ may be obtained from (6) to be

$$\Box\langle F^{0k}\rangle = \left[-\partial^0 J^k + \frac{\partial^k}{\vec{\partial}^2}(\partial^0\partial^m J_m + \Box J^0)\right]\langle\,|\,\rangle \tag{11}$$

where we have taken vacuum expectation values.

We use the action principle (Schwinger (1951a, b, 1953, 1954) and, in particular, Lam (1965); see also Manoukian (1985)) to generate

$$-\mathrm{i}\frac{\delta}{\delta J^{\nu}(x')}\langle F^{0k}(x)\rangle = \langle (A_{\nu}(x')F^{0k}(x))_{+}\rangle - \mathrm{i}\left\langle\left(\frac{\delta}{\delta J^{\nu}(x')}F^{0k}(x)\right)_{+}\right\rangle$$
$$= \langle (A_{\nu}(x')F^{0k}(x))_{+}\rangle - \mathrm{i}g_{\nu}{}^{0}\frac{\partial^{k}}{\vec{\partial}^{2}}\delta(x-x') \quad (12)$$

where we have made use of (9) with its explicit dependence on J^0. The so-called Schwinger term, for $\nu = 0$, on the right-hand side of (12) should be noted and cannot be neglected. We also make use of the basic property of the time-ordered products:

$$\partial_0'\langle (A^m(x')F^{0k}(x))_{+}\rangle = \delta(x^0 - x'^0)\langle [A^m(x'), F^{0k}(x)]\rangle + \langle (\partial_0' A^m(x')F^{0k}(x))_{+}\rangle. \quad (13)$$

Again the commutation relation expression on the right-hand side of (13) cannot be neglected, and upon using the derived equation (10) it yields

$$\partial_0'\langle (A^m(x')F^{0k}(x))_{+}\rangle = \mathrm{i}\left[\frac{\partial^k\partial^m}{\vec{\partial}^2} - \delta^{km}\right]\delta(x-x') + \langle (\partial_0' A^m(x')F^{0k}(x))_{+}\rangle. \quad (14)$$

Equations (11)-(14) then give the following equation for the vacuum expectation value of the time-ordered products of the electric field, in the absence of the external current,

$$\Box\langle (F^{0m}(x')F^{0k}(x))_{+}\rangle = -\mathrm{i}[\vec{\partial}^2\delta^{km} - \partial^k\partial^m]\delta(x-x') \quad (15)$$

with $E^k = F^{0k}$. Similarly, for the magnetic field components F^{mk} we have

$$\Box\langle (F^{rs}(x')F^{mk}(x))_{+}\rangle = \mathrm{i}\partial^m[\partial'^r\delta^{ks} - \partial'^s\delta^{kr}]\delta(x-x') - \mathrm{i}\partial^k[\partial'^r\delta^{ms} - \partial'^s\delta^{mr}]\delta(x-x') \quad (16)$$

with $B^1 = F^{23}$, $B^2 = F^{31}$, $B^3 = F^{12}$.

Equations (15) and (16) are the basic equations to be solved. We now consider two macroscopic neutral perfectly conducting parallel plates, with a separation distance a, placed parallel to the xy plane situated at $z = 0$ and $z = a$, respectively. We also put confining plates at $z = L$ and $z = -L$ (Schwinger 1975) for convenience of the analysis and finally let $L \to \infty$.

The boundary conditions are (cf Jackson (1975) and Hauser (1971) for a lucid discussion of boundary conditions):

$$E^1 = 0 \qquad E^2 = 0 \qquad B^3 = 0 \quad (17)$$

for $z = 0$ and $z = a$.

For example, from (15)

$$\Box\langle (E^1(x')E^1(x))_{+}\rangle = -\mathrm{i}(\partial^2\partial^2 + \partial^3\partial^3)\delta(x-x') \quad (18)$$

and the boundary condition on E^1 in (17) implies a Fourier sine series for $\delta(z-z')$ in (18):

$$\delta(z-z') = \sum_{n=1}^{\infty}\frac{2}{D}\sin n\pi\frac{(z-d)}{D}\sin n\pi\frac{(z'-d)}{D} \quad (19)$$

where $(D = a, d = 0)$ for $0 < z, z' < a$, $(D = L - a, d = a)$ for $a < z, z' < L$, $(D = L, d = 0)$ for $-L < z, z' < 0$, using the completeness relation of the sine functions. On the other hand

$$\Box\langle (E^3(x')E^i(x))_{+}\rangle = \mathrm{i}\partial^3\partial^i\delta(x-x') \qquad i = 1, 2 \quad (20)$$

and the boundary conditions $E^1=0$, $E^2=0$ in (17) imply a Fourier cosine series for $\delta(z-z')$ in (20):

$$\delta(z-z')=\frac{1}{D}+\sum_{n=1}^{\infty}\frac{2}{D}\cos n\pi\frac{(z-d)}{2D}\cos n\pi\frac{(z'-d)}{D} \tag{21}$$

where (D, d) is defined below (19). We then note, in particular, from (20) that

$$\langle(\partial_3' E^3(x')\cdot)_+\rangle=0 \tag{22}$$

for $z'=0$, $z'=a$. Similarly for the magnetic field component B^3 we have from (16)

$$\Box\langle(B^3(x')B^3(x))_+\rangle=-\mathrm{i}(\partial^1\partial^1+\partial^2\partial^2)\delta(x-x') \tag{23}$$

and the boundary condition $B^3=0$ in (17) implies the expansion in (19) for $\delta(z-z')$. Also from (16)

$$\Box\langle(B^i(x')B^3(x))_+\rangle=-\mathrm{i}\partial^i\partial'^3\delta(x-x') \qquad i=1,2 \tag{24}$$

and

$$\langle\partial_3' B^i(x')\cdot)_+\rangle=0 \tag{25}$$

for $z'=0$, $z'=a$.

The energy density T_{00} may be obtained from (7) and (1) to be, in the absence of the external current,

$$\begin{aligned}T_{00}&=(\partial_3^{-1}\partial^0F^{i3})\partial_0A_i+\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\\&=(\partial_3^{-1}\partial^0F^{k3})\partial_0A_k+\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\end{aligned} \tag{26}$$

or

$$T_{00}=\partial^k[(\partial_3^{-1}\partial^0A^3)\partial_0A_k]-\partial^0A_k\partial_0A^k+\tfrac{1}{2}F_{0k}F^{0k}+\tfrac{1}{4}F_{mk}F^{mk}. \tag{27}$$

The total three-dimensional derivative on the right-hand side of (27) cancels out above and below the plates for $L\to\infty$ by the Green theorem, and we may effectively write

$$T_{00}=\tfrac{1}{2}F^{0k}F^{0k}+\tfrac{1}{4}F^{mk}F^{mk} \tag{28}$$

where we have set $A^0=0$ consistent with the field equation (4): $A^0=-(1/\vec{\partial}^2)J^0$.

From (15) and (16) we may write

$$\Box\langle(\boldsymbol{E}_T(x')\cdot\boldsymbol{E}_T(x))_+\rangle=-\mathrm{i}(\vec{\partial}^2+\partial^3\partial^3)\delta(x-x') \tag{29}$$

$$\begin{aligned}\Box\langle(E^3(x')E^3(x))_+\rangle&=-\mathrm{i}(\partial^1\partial^1+\partial^2\partial^2)\delta(x-x')\\&=-\mathrm{i}(\vec{\partial}^2-\partial^3\partial^3)\delta(x-x')\end{aligned} \tag{30}$$

$$\begin{aligned}\Box\langle(B^3(x')B^3(x))_+\rangle&=-\mathrm{i}(\partial^1\partial^1+\partial^2\partial^2)\delta(x-x')\\&=-\mathrm{i}(\vec{\partial}^2-\partial^3\partial^3)\delta(x-x')\end{aligned} \tag{31}$$

$$\Box\langle(\boldsymbol{B}_T(x')\cdot\boldsymbol{B}_T(x))_+\rangle=-\mathrm{i}(\vec{\partial}^2+\partial^3\partial^3)\delta(x-x'). \tag{32}$$

By using the definition from (28):

$$\begin{aligned}\int \mathrm{d}^3\boldsymbol{x}\langle T_{00}(x)\rangle=&\tfrac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}x'^0\,\delta(x^0-x'^0)\langle(\boldsymbol{E}(\boldsymbol{x},x'^0)\cdot\boldsymbol{E}(\boldsymbol{x},x^0))_+\rangle\\&+\tfrac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}x'^0\,\delta(x^0-x'^0)\langle(\boldsymbol{B}(\boldsymbol{x},x'^0)\cdot\boldsymbol{B}(\boldsymbol{x},x^0))_+\rangle\end{aligned} \tag{33}$$

we obtain from (29)-(32) by making use of the analysis given through (17)-(25):

$$\int \mathrm{d}^3\boldsymbol{x}\langle T_{00}(x)\rangle=A\sum_{n=1}^{\infty}\int \mathrm{d}x'^0\,\delta(x^0-x'^0)\int\frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\{\cdot\} \tag{34}$$

$$\{\cdot\}=\left[\boldsymbol{k}^2+\left(\frac{n\pi}{a}\right)^2\right]^{1/2}\exp\left\{-\mathrm{i}\left[\boldsymbol{k}^2+\left(\frac{n\pi}{a}\right)^2\right]^{1/2}|x^0-x'^0|\right\}$$
$$+\left[\boldsymbol{k}^2+\left(\frac{n\pi}{L}\right)^2\right]^{1/2}\exp\left\{-\mathrm{i}\left[\boldsymbol{k}^2+\left(\frac{n\pi}{L}\right)^2\right]^{1/2}|x^0-x'^0|\right\}$$
$$+\left[\boldsymbol{k}^2+\left(\frac{n\pi}{L-a}\right)^2\right]^{1/2}\exp\left\{-\mathrm{i}\left[\boldsymbol{k}^2+\left(\frac{n\pi}{L-a}\right)^2\right]^{1/2}|x^0-x'^0|\right\} \tag{35}$$

where A is the area of each of the macroscopic plates and where we have used the normalisation conditions

$$\frac{2}{D}\int_0^D \mathrm{d}z\,\sin^2\frac{n\pi z}{D}=1=\frac{2}{D}\int_0^D \mathrm{d}z\,\cos^2\frac{n\pi z}{D}. \tag{36}$$

Upon identifying the left-hand side of (34) with the total energy E in the system, we obtain for the Casimir force F by straightforward manipulations

$$F=-\frac{\partial}{\partial a}\left(\frac{E}{A}\right)$$
$$=-\frac{1}{2\pi a}\lim_{L\to\infty}\int \mathrm{d}\tau\frac{\delta(\tau)}{\mathrm{i}|\tau|}\frac{\mathrm{d}^2}{\mathrm{d}|\tau|^2}\left[\left(1-\exp-\frac{\mathrm{i}\pi}{a}|\tau|\right)^{-1}\right.$$
$$\left.-\frac{a}{L-a}\left(1-\exp-\frac{\mathrm{i}\pi}{L-a}|\tau|\right)^{-1}\right]$$
$$=\frac{1}{2\pi a}\int \mathrm{d}\tau\frac{\delta(\tau)}{|\tau|}\frac{\mathrm{d}^2}{\mathrm{d}|\tau|^2}\left(\frac{\mathrm{i}}{(1-\exp-\mathrm{i}(\pi/a)|\tau|]}-\frac{a}{\pi|\tau|}\right) \tag{37}$$

which is exactly twice the result obtained by Schwinger (1975) for a scalar field. The right-hand side of (37) is then readily evaluated (Schwinger 1975) to yield

$$F=-\frac{\pi^2}{240}\frac{1}{a^4} \tag{38}$$

by using the identity and the expansion

$$\frac{\mathrm{i}}{1-\mathrm{e}^{-\mathrm{i}x}}=\tfrac{1}{2}\cot\tfrac{1}{2}x+\tfrac{1}{2}\mathrm{i} \tag{39}$$

$$\cot x=\frac{1}{x}-\frac{1}{3}x-\frac{1}{45}x^3-\ldots. \tag{40}$$

The inclusion of temperature amounts to replacing (Schwinger 1975) the exponentials $\exp(-\mathrm{i}E|\tau|)$ in (35) by

$$\exp(-\mathrm{i}E|\tau|)+\frac{2}{\mathrm{e}^{\beta E}-1}\cos E|\tau| \tag{41}$$

where $\beta=1/kT$ and this is worked out in detail in Schwinger (1975) and will not be repeated here, giving the experimentally difficult to verify Lifschitz (1956) result:

$$F=-\pi\frac{1}{\beta a^3}\sum_{n=1}^{\infty}n^2\ln[1-\exp(-n\pi\beta/a)]-\frac{\pi^2}{45}\frac{1}{\beta^4} \tag{42}$$

with a high-temperature limit

$$F \simeq -\frac{1}{4\pi\beta a^3}\zeta(3) \tag{43}$$

where $\zeta(n)$ is the Riemann zeta function.

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